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Strong converse inequalities for Baskakov operators ☆

Shunsheng Guo and Qiulan Qi*

College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050016, People's Republic of China

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Abstract

We give a strong converse inequality of type B in terms of unified K-functional $K_\lambda^\alpha(f,t^2)(0\leqslant\lambda\leqslant 1,\ 0<\alpha<2)$ for Baskakov operators.

MSC: 41A15; 41A10

Keywords: Baskakov operator; K-functional; Strong converse inequality

1. Introduction

The direct and converse inequalities in the sup-norm for positive linear operators, given in terms of higher-order Ditzian–Totik modulus of smoothness have been widely discussed. For the Bernstein polynomials [1]:

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

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*Corresponding author.
E-mail addresses: ssguo@hebtu.edu.cn (S. Guo), qiqiulan@163.com (Q. Qi).

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Ditzian [1] gave an interesting direct estimate

$$|B_n(f,x)-f(x)|\leqslant C\omega^2_{\varphi^\lambda}\left(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right),\quad 0\leqslant\lambda\leqslant 1,\ \ \varphi(x)=\sqrt{x(1-x)},$$

where $\omega^2_{\varphi^\lambda}(f,t)=\sup_{0<h\leqslant t}\sup_{x\pm h\varphi^\lambda(x)\in[0,1]}|\Delta^2_{h\varphi^\lambda}f(x)|$, which unified the classical estimate for $\lambda=0$ and the norm estimate for $\lambda=1$. Using the higher-order weighted modulus of smoothness $\omega^{2r}_{\varphi^\lambda}(f,t\ (0\leqslant\lambda\leqslant 1)$, in [5] we got the direct, converse and equivalent theorems for the linear combinations of Bernstein operators.

As the inverse results, E. Van Wickeren proved in [10]

$$\omega^2_\alpha\left(f,\frac{1}{\sqrt{n}}\right)\leqslant Cn^{-1}\sum_{k=1}^{n}||B_kf-f||_\alpha\quad(0\leqslant\alpha\leqslant 2),\tag{1.1}$$

where $\omega^2_\alpha(f,t)=\sup\{\varphi^{-\alpha}(x)|\Delta^2_{h\varphi(x)}f(x)|:x,x\pm h\varphi(x)\in[0,1],0<h\leqslant t\}$, $\varphi(x)=\sqrt{x(1-x)}$, $\Delta^2_{h\varphi}f(x)=f(x+h\varphi(x))-2f(x)+f(x-h\varphi(x))$, $||f||_\alpha=||\varphi^{-\alpha}f||_{C[0,1]}$. But, this is only a norm estimate (with $\omega^2_\varphi(f,t)$), not including of the classical one (with $\omega^2(f,t)$).

Ditzian and Ivanov [2] gave the strong converse inequality of type B for the Bernstein operator, which means that there exists a constant K such that

$$\omega^2_\varphi(f,\frac{1}{\sqrt{n}})\leqslant C(||B_nf-f||_{C[0,1]}+||B_{Kn}f-f||_{C[0,1]}),\tag{1.2}$$

where $\omega^2_\varphi(f,t)=\sup_{0<h\leqslant t}||\Delta^2_{h\varphi}f||$, $\varphi(x)=\sqrt{x(1-x)}$.

Totik [9] extended the Ditzian–Ivanov result to a large family of operators. With regard to strong converse inequalities of type A, we reference Totik's paper [8] dealing with the L_∞-norm for the Bernstein, Szász and Baskakov operators, and the work by Gonska and Zhou [4] concerning the L_p-norm $(1<p\leqslant\infty)$ for Bernstein–Kantorovich operators.

For $f\in C[0,\infty)$ (the set of bounded and continuous functions), the Baskakov operators are defined by

$$V_n(f,x)=\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)V_{n,k}(x),\quad V_{n,k}(x)=\binom{n+k-1}{k}x^k(1+x)^{-n-k}.$$

In [6], we obtained a Stechkin–Marchaud-type inequality similar to (1.1) for Baskakov operator, there, we failed getting a result of type (1.2).

To state our results of this paper, we give some notations (cf. [6]):

For $0\leqslant\lambda\leqslant 1,\ 0<\alpha<2,\ \varphi^2(x)=x(1+x)$, we write

$$C_0:=\{f\in C[0,\infty):f(0)=0\},\quad C^2:=\{f\in C_0:f''\in C[0,\infty)\},$$

$$||f||_0^*:=\sup_{x\in(0,\infty)}|\varphi^{\alpha(\lambda-1)}(x)f(x)|,\quad ||f||_2^*:=\sup_{x\in[0,\infty)}|\varphi^{2+\alpha(\lambda-1)}(x)f''(x)|,$$

$$C^0_{\lambda,\alpha} := \{f \in C_0, ||f||_0^* < \infty\}, \quad C^2_{\lambda,\alpha} := \{f \in C^2, ||f||_2^* < \infty\},$$

$$K_\lambda^\alpha(f, t^2) := \inf_{g \in C^2_{\lambda,\alpha}} \{||f-g||_0^* + t^2||g||_2^*\}.$$

With the K-functional $K_\lambda^\alpha(f,t^2)(0 \leqslant \lambda \leqslant 1, 0<\alpha<2)$, we obtain the strong converse inequality of type (1.2) for Baskakov operators:

Theorem. *Suppose* $0 \leqslant \lambda \leqslant 1, 0<\alpha<2, f \in C^0_{\lambda,\alpha}$, *there exists a constant* $K>1$, *for* $l \geqslant Kn$, *we have*

$$K_\lambda^\alpha\left(f, \frac{1}{n}\right) \leqslant C\frac{l}{n}(||V_nf-f||_0^* + ||V_lf-f||_0^*).$$

Throughout this paper, C denotes a constant independent of n and x, but it is not necessarily the same in different cases.

2. Lemmas

In order to prove our main result, we give some fundamental lemmas.

Lemma 2.1 (Ditzian and Totik [3]). *For* $x \in [0, \infty)$, *we have*

$$V_n(1,x) = 1, \quad V_n(t-x,x) = 0, \quad V_n((t-x)^2,x) = \frac{\varphi^2(x)}{n},$$

$$V_n((t-x)^4,x) = \frac{3\varphi^4(x)}{n^2} + \frac{\varphi^2(x)}{n^3}((1+2x)^2 + 2\varphi^2(x)) \leqslant C\left(\frac{\varphi^4(x)}{n^2} + \frac{\varphi^2(x)}{n^3}\right).$$

Proof. From the recursion relation ([3] (9.4.13)), by simple calculations, we can obtain these results about moments. □

Lemma 2.2.

$$\sum_{k=0}^{\infty} \varphi^{-4}\left(\frac{k+1}{n}\right) V_{n+2,k}(x) \leqslant 7\varphi^{-4}(x), \quad (n \geqslant 3), \tag{2.1}$$

$$\sum_{k=0}^{\infty} \frac{n^3}{(n+k)^3} V_{n,k}(x) \leqslant 11(1+x)^{-3}, \quad (n \geqslant 4). \tag{2.2}$$

Proof. For $n\geqslant 3$, by simple computation, we get

$$\begin{aligned}&\sum_{k=0}^{\infty}\varphi^{-4}\left(\frac{k+1}{n}\right)V_{n+2,k}(x)\\&\quad=\frac{1}{x^2(1+x)^2}\sum_{k=3}^{\infty}\frac{(k+2)n^4(n+k)}{(k+1)(n+k+1)(n+1)n(n-1)(n-2)}V_{n-2,k+2}(x)\\&\quad\leqslant\frac{7}{x^2(1+x)^2}\sum_{k=3}^{\infty}V_{n-2,k+2}(x)\leqslant 7\varphi^{-4}(x).\end{aligned}$$

That is (2.1).

Similarly, for $n\geqslant 4$, we can get (2.2) by the relation

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{n^3}{(n+k)^3}V_{n,k}(x)\\&\quad=(1+x)^{-3}\sum_{k=4}^{\infty}\frac{n^3}{(n+k)^3}\cdot\frac{(n+k-1)(n+k-2)(n+k-3)}{(n-1)(n-2)(n-3)}V_{n-3,k}(x)\\&\quad\leqslant 11(1+x)^{-3}.\quad\square\end{aligned}$$

Lemma 2.3. *For* $r\in N, -2r<\mu<0, x\in(0,\infty), t\in[0,\infty)$, *we have*

$$\left|\int_x^t |t-u|^{r-1}\varphi^{\mu}(u)\,du\right|\leqslant C|t-x|^r(\varphi^{\mu}(x)+x^{\frac{\mu}{2}}(1+t)^{\frac{\mu}{2}}).$$

Proof. Let $u=t+\tau(x-t),\ 0\leqslant\tau\leqslant 1$, then

$$\begin{aligned}\left|\int_x^t |t-u|^{r-1}\varphi^{\mu}(u)\,du\right|&\leqslant\left|\int_x^t |t-u|^{r-1}u^{\mu/2}\,du\right|\cdot((1+x)^{\frac{\mu}{2}}+(1+t)^{\frac{\mu}{2}})\\&\leqslant\int_0^1\tau^{r-1}|t-x|^r\cdot(\tau x+(1-\tau)t)^{\mu/2}\,d\tau((1+x)^{\frac{\mu}{2}}+(1+t)^{\frac{\mu}{2}})\\&\leqslant|t-x|^r\cdot\int_0^1\tau^{r+\mu/2-1}x^{\mu/2}\,d\tau\cdot((1+x)^{\frac{\mu}{2}}+(1+t)^{\frac{\mu}{2}})\\&\leqslant\frac{1}{r+\mu/2}\cdot|t-x|^r(\varphi^{\mu}(x)+x^{\frac{\mu}{2}}(1+t)^{\frac{\mu}{2}}).\quad\square\end{aligned}$$

Lemma 2.4. *For* $0\leqslant\lambda\leqslant 1,\ 0<\alpha<2,\ f\in C^2_{\lambda,\alpha},\ n\geqslant 3$, *we have*

$$||\varphi^3V_n'''f||_0^*\leqslant C\sqrt{n}||f||_2^*, \tag{2.3}$$

and

$$||\varphi^2V_n'''f||_0^*\leqslant Cn||f||_2^*. \tag{2.4}$$

Proof. Using the relations $\varphi^2(x)V'_{n+2,k}(x) = (n+2)(\frac{k}{n+2} - x)V_{n+2,k}(x)$ and $V''_n(f,x) = n(n+1)\sum_{k=0}^{\infty} V_{n+2,k}(x)\overrightarrow{\Delta}^2_{\frac{1}{n}} f(\frac{k}{n})$, we have

$$
\begin{aligned}
&|\varphi^{3+\alpha(\lambda-1)}(x)V'''_n(f,x)| \\
&\quad = \left| n(n+1)(n+2)\varphi^{1+\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty}\left(\frac{k}{n+2} - x\right)V_{n+2,k}(x) \right. \\
&\qquad \left. \times \int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} f''\left(\frac{k}{n} + s + t\right) ds\, dt \right| \\
&\quad \leqslant ||f||_2^* \cdot \varphi^{1+\alpha(\lambda-1)}(x)n(n+1)(n+2) \\
&\qquad \times \sum_{k=0}^{\infty}\left|\frac{k}{n+2} - x\right| V_{n+2,k}(x) \cdot \int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} \varphi^{-2-\alpha(\lambda-1)}\left(\frac{k}{n} + s + t\right) ds\, dt.
\end{aligned}
$$

Using the Hölder inequality and the relation (cf. [7]):

$$
\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} \varphi^{-2}\left(\frac{k}{n} + u + v\right) du\, dv \leqslant \frac{C}{n(n+1)}\varphi^{-2}\left(\frac{k+1}{n}\right),
$$

we can get

$$
\begin{aligned}
&\left| \int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} \varphi^{-2-\alpha(\lambda-1)}\left(\frac{k}{n} + s + t\right) ds\, dt \right| \\
&\quad \leqslant \left(\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} \varphi^{-2}\left(\frac{k}{n} + s + t\right) ds\, dt\right)^{\frac{2+\alpha(\lambda-1)}{2}} \cdot \left(\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} ds\, dt\right)^{1-\frac{2+\alpha(\lambda-1)}{2}} \\
&\quad \leqslant n^{\alpha(\lambda-1)} \cdot \left(\frac{C}{n(n+1)}\varphi^{-2}\left(\frac{k+1}{n}\right)\right)^{\frac{2+\alpha(\lambda-1)}{2}} \\
&\quad = Cn^{-2}\varphi^{-2-\alpha(\lambda-1)}\left(\frac{k+1}{n}\right),
\end{aligned}
\tag{2.5}
$$

thus, by the Hölder inequality and Lemmas 2.1, 2.2, we can deduce,

$$
\begin{aligned}
&|\varphi^{3+\alpha(\lambda-1)}(x)V'''_n(f,x)| \leqslant ||f||_2^*\varphi^{1+\alpha(\lambda-1)}(x)n(n+1)(n+2) \\
&\qquad \times \left(\sum_{k=0}^{\infty}\left(\frac{k}{n+2} - x\right)^2 V_{n+2,k}(x)\right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{k=0}^{\infty}\left(\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}} \varphi^{-2-\alpha(\lambda-1)}\left(\frac{k}{n} + s + t\right) ds\, dt\right)^2 V_{n+2,k}(x)\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leqslant C\|f\|_2^* n(n+1)(n+2)\left(\frac{\varphi^2(x)}{n+2}\right)^{1/2}\varphi^{1+\alpha(\lambda-1)}(x)\cdot\frac{1}{n^2}$$
$$\times\left(\sum_{k=0}^{\infty}\varphi^{-4-2\alpha(\lambda-1)}\left(\frac{k+1}{n}\right)V_{n+2,k}(x)\right)^{\frac{1}{2}}$$
$$\leqslant C\|f\|_2^*\sqrt{n}\varphi^{2+\alpha(\lambda-1)}(x)\left(\sum_{k=0}^{\infty}\varphi^{-4}\left(\frac{k+1}{n}\right)V_{n+2,k}(x)\right)^{\frac{2+\alpha(\lambda-1)}{4}}$$
$$\leqslant C\sqrt{n}\|f\|_2^*.$$

This is (2.3). Next we prove (2.4).

By [3, (9.4.3)] we have

$$|\varphi^{2+\alpha(\lambda-1)}(x)V_n'''(f,x)|$$
$$= |n(n+1)(n+2)\varphi^{2+\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty}\vec{\Delta}^3_{\frac{1}{n}}f\left(\frac{k}{n}\right)V_{n+3,k}(x)|$$
$$\leqslant Cn^3\varphi^{2+\alpha(\lambda-1)}(x)\left(\sum_{k=0}^{\infty}\left|\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}f''\left(\frac{k+1}{n}+u+v\right)du\,dv\right|V_{n+3,k}(x)\right.$$
$$\left.+\sum_{k=0}^{\infty}\left|\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}f''\left(\frac{k}{n}+u+v\right)du\,dv\right|V_{n+3,k}(x)\right)$$
$$=: Cn^3\varphi^{2+\alpha(\lambda-1)}(x)(I_1+I_2).$$

From (2.5) and the procedure of the proof of (2.1), we get

$$I_1\leqslant \|f\|_2^*\sum_{k=0}^{\infty}\left|\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\varphi^{-2-\alpha(\lambda-1)}\left(\frac{k+1}{n}+u+v\right)du\,dv\right|V_{n+3,k}(x)$$
$$\leqslant C\|f\|_2^* n^{-2}\sum_{k=0}^{\infty}\varphi^{-2-\alpha(\lambda-1)}\left(\frac{k+2}{n}\right)V_{n+3,k}(x)\leqslant C\|f\|_2^* n^{-2}\varphi^{-2-\alpha(\lambda-1)}(x).$$
$$I_2\leqslant \|f\|_2^*\sum_{k=0}^{\infty}\left|\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\varphi^{-2-\alpha(\lambda-1)}\left(\frac{k}{n}+u+v\right)du\,dv\right|V_{n+3,k}(x)$$
$$\leqslant C\|f\|_2^* n^{-2}\sum_{k=0}^{\infty}\varphi^{-2-\alpha(\lambda-1)}\left(\frac{k+1}{n}\right)V_{n+3,k}(x)\leqslant C\|f\|_2^* n^{-2}\varphi^{-2-\alpha(\lambda-1)}(x).$$

Hence we have

$$|\varphi^{2+\alpha(\lambda-1)}(x)V_n'''(f,x)|\leqslant Cn\|f\|_2^*.$$

This is (2.4). □

Lemma 2.5. *For* $0\leqslant\lambda\leqslant 1,\ 0<\alpha<2, n\geqslant 4,\ f^{(i)}(x)\in C^0_{\lambda,\alpha},\ i=0,1,2,3, \varphi^3 f'''\in C^0_{\lambda,\alpha}$, *we have*

$$\left\|V_n f - f - \frac{\varphi^2 f''}{2n}\right\|_0^* \leqslant C(n^{-\frac{3}{2}}\|\varphi^3 f'''\|_0^* + n^{-2}\|\varphi^2 f'''\|_0^*). \tag{2.6}$$

Proof. We expand $f(t)$ by the Taylor formula

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \frac{1}{2}\int_x^t (t-v)^2 f'''(v)\,dv,$$

and use Lemma 2.1 to obtain

$$\varphi^{\alpha(\lambda-1)}(x)\left(V_n(f,x) - f(x) - \frac{\varphi^2(x)f''(x)}{2n}\right) = I_n(f,x), \tag{2.7}$$

where $I_n(f,x) = \frac{1}{2}\varphi^{\alpha(\lambda-1)}(x)V_n(\int_x^t (t-v)^2 f'''(v)\,dv, x)$.

By [3, (9.4.14)], for $x\in E_n = [\frac{1}{n},\infty)$, we have

$$V_n((t-x)^{2m},x)\leqslant C\frac{\varphi^{2m}(x)}{n^m}. \tag{2.8}$$

By Lemma 2.1, for $x\in E_n^c = [0,\frac{1}{n})$, we have

$$V_n((t-x)^4,x)\leqslant C\frac{\varphi^2(x)}{n^3}. \tag{2.9}$$

Next we will estimate (2.7) in two cases: $x\in E_n$ and $x\in E_n^c$.

For $x\in E_n$, by (2.8) and Lemma 2.3, we can get

$$\begin{aligned}|I_n(f,x)| &\leqslant \frac{1}{2}\varphi^{\alpha(\lambda-1)}(x)\cdot\sum_{k=0}^{\infty}\left|\int_x^{\frac{k}{n}}\left(\frac{k}{n}-v\right)^2 f'''(v)\,dv\right|V_{n,k}(x)\\ &\leqslant \|\varphi^3 f'''\|_0^*\varphi^{\alpha(\lambda-1)}(x)\cdot\sum_{k=0}^{\infty}\left|\int_x^{\frac{k}{n}}\frac{|\frac{k}{n}-v|^2}{\varphi^{3+\alpha(\lambda-1)}(v)}\,dv\right|V_{n,k}(x)\\ &\leqslant \|\varphi^3 f'''\|_0^*\varphi^{\alpha(\lambda-1)}(x)\cdot\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^3\\ &\quad\times\left[\varphi^{-3-\alpha(\lambda-1)}(x) + x^{-\frac{3+\alpha(\lambda-1)}{2}}\left(1+\frac{k}{n}\right)^{-\frac{3+\alpha(\lambda-1)}{2}}\right]V_{n,k}(x).\end{aligned} \tag{2.10}$$

Using (2.8) we get

$$\begin{aligned}&\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^{3}\varphi^{-3-\alpha(\lambda-1)}(x)V_{n,k}(x)\\&\leqslant\varphi^{-3-\alpha(\lambda-1)}(x)\left(\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^{6}V_{n,k}(x)\right)^{\frac{1}{2}}\\&\leqslant C\varphi^{-3-\alpha(\lambda-1)}(x)\frac{\varphi^{3}(x)}{n^{\frac{3}{2}}}=Cn^{-\frac{3}{2}}\varphi^{-3-\alpha(\lambda-1)}(x),\end{aligned}\tag{2.11}$$

and by Lemma 2.2, we have

$$\begin{aligned}&\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^{3}x^{-\frac{3+\alpha(\lambda-1)}{2}}\left(1+\frac{k}{n}\right)^{-\frac{3+\alpha(\lambda-1)}{2}}V_{n,k}(x)\\&\leqslant x^{-\frac{3+\alpha(\lambda-1)}{2}}\left(\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^{6}V_{n,k}(x)\right)^{\frac{1}{2}}\cdot\left(\sum_{k=0}^{\infty}\left(1+\frac{k}{n}\right)^{-3-\alpha(\lambda-1)}V_{n,k}(x)\right)^{\frac{1}{2}}\\&\leqslant Cx^{-\frac{3+\alpha(\lambda-1)}{2}}\frac{\varphi^{3}(x)}{n\sqrt{n}}\cdot\left(\sum_{k=0}^{\infty}\left(\frac{n}{n+k}\right)^{3}V_{n,k}(x)\right)^{\frac{3+\alpha(\lambda-1)}{6}}\\&\leqslant Cx^{-\frac{3+\alpha(\lambda-1)}{2}}\frac{\varphi^{3}(x)}{n\sqrt{n}}\cdot(1+x)^{-\frac{3+\alpha(\lambda-1)}{2}}=Cn^{-\frac{3}{2}}\varphi^{-\alpha(\lambda-1)}(x).\end{aligned}\tag{2.12}$$

From (2.10)–(2.12) we obtain for $x\in E_n$

$$|I_n(f,x)|\leqslant Cn^{-\frac{3}{2}}||\varphi^{3}f'''||_0^{*}.\tag{2.13}$$

For $x\in E_n^c$, by Lemma 2.3 we have

$$\begin{aligned}|I_n(f,x)|&\leqslant\frac{1}{2}\varphi^{\alpha(\lambda-1)}(x)\cdot\sum_{k=0}^{\infty}\left|\int_x^{\frac{k}{n}}\left(\frac{k}{n}-v\right)^{2}f'''(v)\,dv\right|V_{n,k}(x)\\&\leqslant||\varphi^{2}f'''||_0^{*}\varphi^{\alpha(\lambda-1)}(x)\cdot\sum_{k=0}^{\infty}\left|\int_x^{\frac{k}{n}}\frac{|\frac{k}{n}-v|^{2}}{\varphi^{2+\alpha(\lambda-1)}(v)}\,dv\right|V_{n,k}(x)\\&\leqslant||\varphi^{2}f'''||_0^{*}\varphi^{\alpha(\lambda-1)}(x)\cdot\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^{3}\cdot\left[\varphi^{-2-\alpha(\lambda-1)}(x)\right.\\&\qquad\left.+x^{-\frac{2+\alpha(\lambda-1)}{2}}\left(1+\frac{k}{n}\right)^{-\frac{2+\alpha(\lambda-1)}{2}}\right]V_{n,k}(x).\end{aligned}$$

Note that for $x \in E_n^c, \varphi(x) \sim \sqrt{x}$, so we have for $x \in E_n^c$

$$\varphi^{-2-\alpha(\lambda-1)}(x) + \left(x\left(1+\frac{k}{n}\right)\right)^{-\frac{2+\alpha(\lambda-1)}{2}} \leqslant \varphi^{-2-\alpha(\lambda-1)}(x) + x^{-\frac{2+\alpha(\lambda-1)}{2}} \leqslant 2\varphi^{-2-\alpha(\lambda-1)}(x).$$

Hence we get for $x \in E_n^c$ by (2.9)

$$\begin{aligned} |I_n(f,x)| &\leqslant C\varphi^{\alpha(\lambda-1)}(x)\varphi^{-2-\alpha(\lambda-1)}(x)||\varphi^2 f'''||_0^* \sum_{k=0}^{\infty}\left|\frac{k}{n}-v\right|^3 V_{n,k}(x) \\ &\leqslant C\varphi^{-2}(x)||\varphi^2 f'''||_0^*\left(\sum_{k=0}^{\infty}\left|\frac{k}{n}-v\right|^2 V_{n,k}(x)\right)^{\frac{1}{2}}\cdot\left(\sum_{k=0}^{\infty}|\frac{k}{n}-v|^4 V_{n,k}(x)\right)^{\frac{1}{2}} \\ &\leqslant Cn^{-2}||\varphi^2 f'''||_0^*. \end{aligned} \tag{2.14}$$

Combining (2.13) and (2.14), we obtain

$$|I_n(f,x)| \leqslant C(n^{-3/2}||\varphi^3 f'''||_0^* + n^{-2}||\varphi^2 f'''||_0^*).$$

Thus we have proved (2.6).

Lemma 2.6 (Guo et al. [6, (2.1)]). *For $0 \leqslant \lambda \leqslant 1,\ 0 < \alpha < 2, f \in C_{\lambda,\alpha}^0$, we have*

$$||V_n f||_2^* \leqslant Cn||f||_0^*.$$

Lemma 2.7 (Guo et al. [6, Lemma 2.5]). *For $0 \leqslant \gamma \leqslant 2, t > 0, x \geqslant t$, and either of*

(i) $0 < t < 1$,
(ii) $x \geqslant 2t$

is satisfied, then we have

$$\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{-\frac{t}{2}}^{\frac{t}{2}} \varphi^{-\gamma}(x+u+v)\,du\,dv \leqslant C(\gamma)t^2\varphi^{-\gamma}(x).$$

Lemma 2.8. *For $f \in C_{\lambda,\alpha}^0$, we have*

$$||V_n f||_0^* \leqslant 2||f||_0^*.$$

Proof. By the relations $V_n(t,x) = x$, $V_n(t^2,x) = \frac{\varphi^2(x)}{n} + x^2$ and the Hölder inequality, we can write

$$\left|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty} V_{n,k}(x) f\left(\frac{k}{n}\right)\right| \leqslant \|f\|_0^* \cdot \left|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty} V_{n,k}(x)\varphi^{\alpha(1-\lambda)}\left(\frac{k}{n}\right)\right|$$
$$\leqslant \|f\|_0^* \cdot \varphi^{\alpha(\lambda-1)}(x)\left(\sum_{k=0}^{\infty} V_{n,k}(x)\varphi^2\left(\frac{k}{n}\right)\right)^{\frac{\alpha(1-\lambda)}{2}}$$
$$\leqslant 2\|f\|_0^*.$$

3. Main results

With the lemmas in Section 2, we will prove our main results in this section.

Theorem 3.1. *Suppose* $0 \leqslant \lambda \leqslant 1,\ 0<\alpha<2, n\geqslant 4, f\in C^0_{\lambda,\alpha}$, *there exists a constant* $K>1$, *for* $l \geqslant Kn$, *we have*

$$K_\lambda^\alpha\left(f,\frac{1}{n}\right) \leqslant C\frac{l}{n}(\|V_n f - f\|_0^* + \|V_l f - f\|_0^*).$$

Proof. Using the definition of $K_\lambda^\alpha(f,\frac{1}{n})$ and Lemma 2.8, we have

$$K_\lambda^\alpha\left(f,\tfrac{1}{n}\right) \leqslant \|f - V_n^2 f\|_0^* + \tfrac{1}{n}\|V_n^2 f\|_2^* \leqslant 2\|V_n f - f\|_0^* + \tfrac{1}{n}\|V_n^2 f\|_2^*. \tag{3.1}$$

From Lemma 2.5, we have

$$\left\|V_l(V_n^2 f) - V_n^2 f - \frac{\varphi^2 (V_n^2 f)''}{2l}\right\|_0^* \leqslant C(l^{-\frac{3}{2}}\|\varphi^3(V_n^2 f)'''\|_0^* + l^{-2}\|\varphi^2(V_n^2 f)'''\|_0^*),$$

therefore, combining Lemmas 2.4, 2.6, 2.8, we get

$$\frac{1}{2l}\|V_n^2 f\|_2^* \leqslant \|V_l(V_n^2 f) - V_n^2 f\|_0^* + Cl^{-\frac{3}{2}}\|\varphi^3(V_n^2 f)'''\|_0^* + Cl^{-2}\|\varphi^2(V_n^2 f)'''\|_0^*$$
$$\leqslant \|V_l(V_n^2 f) - V_n^2 f\|_0^* + Cl^{-\frac{3}{2}}\sqrt{n}\|V_n f\|_2^* + Cl^{-2}n\|V_n f\|_2^*$$
$$= \|V_l(V_n^2 f - V_n f) + V_l(V_n f - f) + V_l f - f + f - V_n f + V_n f - V_n^2 f\|_0^*$$
$$+ C(l^{-\frac{3}{2}}\sqrt{n} + l^{-2}n)\|V_n f - V_n^2 f + V_n^2 f\|_2^*$$
$$\leqslant C(\|V_l f - f\|_0^* + \|V_n f - f\|_0^* + (l^{-\frac{3}{2}}n^{\frac{3}{2}} + l^{-2}n^2)\|V_n f - f\|_0^*$$
$$+ (l^{-\frac{3}{2}}\sqrt{n} + l^{-2}n)\|V_n^2 f\|_2^*).$$

For $l \geqslant Kn$, we can choose $K>1$ such that $C(l^{-\frac{3}{2}}\sqrt{n} + l^{-2}n) \leqslant \frac{1}{4l}$.

Then, $\frac{1}{4l}\|V_n^2 f\|_2^* \leqslant C(\|V_l f - f\|_0^* + \|V_n f - f\|_0^*$. Therefore, (3.1) implies Theorem 3.1. □

Theorem 3.2. *Under the condition of Theorem* 3.1, *one has*

$$|\varphi^{\alpha(\lambda-1)}(x)\Delta^2_{h\varphi^\lambda} f(x)| \leqslant CK^\alpha_\lambda(f, h^2\varphi^{2(\lambda-1)}(x)),$$

where $\Delta^2_{h\varphi^\lambda} f(x) = f(x + h\varphi^\lambda(x)) - 2f(x) + f(x - h\varphi^\lambda(x))$.

Proof. According to the definition of $K^\alpha_\lambda(f, t^2)$, for fixed x, λ, α, there exists $g_{x,\lambda,\alpha} := g \in C^2_{\lambda,\alpha}$ such that

$$\|f - g\|_0^* + h^2\varphi^{2(\lambda-1)}(x)\|g\|_2^* \leqslant 2K^\alpha_\lambda(f, h^2\varphi^{2(\lambda-1)}(x)). \tag{3.2}$$

On the other hand,

$$|\Delta^2_{h\varphi^\lambda} f(x)| \leqslant |\Delta^2_{h\varphi^\lambda}(f - g)(x)| + |\Delta^2_{h\varphi^\lambda} g(x)|. \tag{3.3}$$

For the first summand of (3.3), since $\varphi^{\alpha(1-\lambda)}(x)$ is a monotone increasing function and $x \geqslant h\varphi^\lambda(x)$, $0 \leqslant \alpha(1-\lambda) < 2$, we have

$$\begin{aligned}
|\Delta^2_{h\varphi^\lambda}(f - g)(x)| &= |(f - g)(x + h\varphi^\lambda(x)) - 2(f - g)(x) + (f - g)(x - h\varphi^\lambda(x))| \\
&\leqslant \|f - g\|_0^* \cdot |\varphi^{\alpha(1-\lambda)}(x + h\varphi^\lambda(x)) + 2\varphi^{\alpha(1-\lambda)}(x) \\
&\quad + \varphi^{\alpha(1-\lambda)}(x - h\varphi^\lambda(x))| \\
&\leqslant \|f - g\|_0^* \cdot |\varphi^{\alpha(1-\lambda)}(2x) + 3\varphi^{\alpha(1-\lambda)}(x)| \\
&\leqslant 7\varphi^{\alpha(1-\lambda)}(x) \cdot \|f - g\|_0^*.
\end{aligned} \tag{3.4}$$

For the second summand of (3.3), for $x \geqslant h\varphi^\lambda(x)$, by Lemma 2.7, we have

$$\begin{aligned}
|\Delta^2_{h\varphi^\lambda} g(x)| &= \left| \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} g''(x + u + v)\, du\, dv \right| \\
&\leqslant \|g\|_2^* \cdot \left| \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}} \varphi^{-2+\alpha(1-\lambda)}(x + u + v)\, du\, dv \right| \\
&\leqslant C\|g\|_2^* h^2\varphi^{(\alpha-2)(1-\lambda)}(x).
\end{aligned} \tag{3.5}$$

Combining (3.3)–(3.5), we have

$$|\varphi^{\alpha(\lambda-1)}(x)\Delta^2_{h\varphi^\lambda} f(x)| \leqslant C(\|f - g\|_0^* + h^2\varphi^{2(\lambda-1)}(x)\|g\|_2^*) \leqslant 2CK^\alpha_\lambda(f, h^2\varphi^{2(\lambda-1)}(x)).$$

The proof is complete. □

Corollary 3.1. *Let* $\lambda = 1,\ f \in C[0,\infty)$, *there exist a constant* $K > 1$ *such that*

$$\omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) \leqslant C(||V_n f - f||_{C[0,\infty)} + ||V_{Kn} f - f||_{C[0,\infty)}), \tag{3.6}$$

here $\omega_\varphi^2(f,t) = \sup_{0<h\leqslant t}\sup_{x\pm h\varphi(x)\in[0,\infty)}|\Delta_{h\varphi}^2 f(x)|$.

Proof. For $\lambda = 1, K_\lambda^\alpha(f,t^2)$ is the usual K-functional $K_\varphi(f,t^2) = \inf_g\{||f - g||_{C[0,\infty)} + t^2||\varphi^2 g''||_{C[0,\infty)}, g' \in A.C._{\text{loc}}\}$, which is equivalent to $\omega_\varphi^2(f,t)$ (cf. [3]). One immediately obtains (3.6) from Theorem 3.1. □

Corollary 3.2. *For* $0<\alpha<2, 0\leqslant\lambda\leqslant 1, f\in C[0,\infty)$, *we have*

$$|V_n(f,x) - f(x)| = O\left(n^{-\frac{\alpha}{2}}\varphi^{\alpha(1-\lambda)}(x)\right) \Rightarrow \omega_{\varphi^\lambda}^2(f,t) = O(t^\alpha),$$

here $\omega_{\varphi^\lambda}^2(f,t) = \sup_{0<h\leqslant t}\sup_{x\pm h\varphi^\lambda(x)\in[0,\infty)}|\Delta_{h\varphi^\lambda}^2 f(x)|$.

Proof. From the condition $|V_n(f,x) - f(x)| = O((n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^\alpha)$ and Theorem 3.1, we have $K_\lambda^\alpha(f,\frac{1}{n}) = O(n^{-\frac{\alpha}{2}})$. For $t: 0<t<1$, we can choose $n\in N$ such that $\frac{1}{\sqrt{n+1}}<t\leqslant\frac{1}{\sqrt{n}}$, then $K_\lambda^\alpha(f,t^2)\leqslant K_\lambda^\alpha(f,n^{-1})\leqslant Ct^\alpha$. By Theorem 3.2, we get

$$|\varphi^{\alpha(\lambda-1)}(x)\Delta_{h\varphi^\lambda}^2 f(x)| \leqslant CK_\lambda^\alpha(f, h^2\varphi^{2(\lambda-1)}(x)) \leqslant Ch^\alpha\varphi^{\alpha(\lambda-1)}(x).$$

That is $\omega_{\varphi^\lambda}^2(f,t) = O(t^\alpha)$. □

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